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The M2-brane Solution of Heterotic M-theory with the Gauss-Bonnet R^2 terms

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Abstract

We consider the effective action of M-theory compactified on a S^1/Z_2 orbifold with R^2 interaction in the Gauss-Bonnet combination. We derive equations of motion with source terms arising from the Gauss-Bonnet terms and find the M2-brane solution up to order $\kappa^{2/3}$. It receives a correction which depends on the orbifold coordinate in the same form as the gauge 5-brane solution.

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1 Introduction

Several years ago, Hořava and Witten [1, 2] showed that the strong coupling limit of ten-dimensional $E_8 \times E_8$ heterotic string theory is described by M-theory compactified on $M^{10} \times S^1/Z_2$ with a set of E_8 gauge fields on two ten-dimensional orbifold fixed planes. The low-energy effective action of this heterotic M-theory consists of two parts, $S_{\rm SG}$ and $S_{\rm YM}$. $S_{\rm SG}$ is the action of usual eleven-dimensional supergravity in the bulk, while $S_{\rm YM}$ is that of super Yang-Mills theory with E_8 gauge fields on the orbifold planes. It is significant to investigate a classical solution of the effective action for a background of this theory. Many interesting aspects are discussed such as the low-dimensional theory with the Calabi-Yau compactification [3, 4], the gaugino condensation [5, 6] and various new models [7, 8].

This effective action is given by an expansion in eleven-dimensional gravitational constant κ [2]. $S_{\rm SG}$ is zero-th order and $S_{\rm YM}$ is first order in $\kappa^{2/3}$. It is known that, at higher order in κ, we need additional interactions of higher powers of the gauge field F and the curvature R [1, 2, 9]. In particular, there is R^2 interaction in the Gauss-Bonnet combination at order $\kappa^{2/3}$ [4, 9]. It is required by anomaly cancellation and supersymmetry as the analogue of ten-dimensional theory [10]. In many cases, however, a contribution of the Gauss-Bonnet R^2 terms is neglected because it is higher order in derivatives.

In the effective theory without the Gauss-Bonnet terms, the soliton solutions were discussed by Lalak et al.[11] The gauge 5-brane solution was constructed explicitly. This solution has a non-trivial dependence on the orbifold coordinate because of source terms which consist of E_8 gauge fields at order $\kappa^{2/3}$. The x^{11}-dependent part of the solution is regarded as a correction from the strongly coupled heterotic string theory at low-energy limit. The M2-brane and M5-brane solutions are also discussed. These solutions, however, do not receive corrections, in contrast to the gauge 5-brane solution, because source terms vanish by the brane ansatz. In addition, it was shown that they are BPS solutions preserving a quarter of the eleven-dimensional supersymmetry for the M2-brane oriented orthogonal to the orbifold planes as well as the M5-brane oriented parallel to the orbifold planes. It is consistent with the suggestion [1] that the M2-brane wrapping on S^1/Z_2 represents the strongly coupled fundamental heterotic string.

In this paper, we take the Gauss-Bonnet terms into account. Since they consist of the metric, a new contribution to source terms appears in the Einstein equation and it can not be ignored. The effect of the Gauss-Bonnet terms is investigated in some five-dimensional models [12], however there are few discussions on heterotic M-theory. We expect that these investigations can reveal new aspects of the low-dimensional models. To this aim, we consider the M2-brane solution of heterotic M-theory and show that it receives a correction of order $\kappa^{2/3}$ from the Gauss-Bonnet terms.

In what follows, equations of motion are derived with the Gauss-Bonnet terms. They are solved in two asymptotic regions up to order $\kappa^{2/3}$ by an ansatz based on the usual M2-brane solution [13]. We show that it receives a modification of order $\kappa^{2/3}$ which depends on the orbifold coordinate in the same form as the gauge 5-brane case [11]. This modification can be regarded as a gravitational effect of Z_2 singularities from the viewpoint of the eleven-dimensional theory, or the strong coupling correction from the viewpoint of ten-dimensional string theory. Finally we discuss interpretations of the solution. In appendix, we plot the solution numerically as a function of x^{11} and r, the coordinates in the orbifold direction and in the transverse direction to M2-brane, respectively. It confirms that the asymptotic solutions are connected smoothly and they describe a smooth solution of the field equations.

2 Heterotic M-theory

We start with the low-energy effective action of M-theory on $M^{10}\times S^1/Z_2$, namely heterotic M-theory [1, 2], with the Gauss-Bonnet R^2 terms [4, 9]. The bosonic part of the action is given by

$$S = S_{\rm SG} + S_{\rm YM}\ . \tag{2.1}$$

Here $S_{\rm SG}$ is the action of familiar eleven-dimensional supergravity [14] given by

$$S_{\rm SG} = \frac{1}{2\kappa^2}\int_{M^{11}} d^{11}x\Big\{\sqrt{-g}\Big(-R - \frac{1}{24}G_{IJKL}G^{IJKL}\Big) - \frac{\sqrt{2}}{1728}\,\epsilon^{I_1\cdots I_{11}}C_{I_1I_2I_3}G_{I_4\cdots I_7}G_{I_8\cdots I_{11}}\Big\} \tag{2.2}$$

where κ denotes the eleven-dimensional gravitational constant and C_{IJK} denotes an abelian three-form gauge field whose field strength is a four-form given by $G_{IJKL} \equiv 4!\,\partial_{[I}C_{JKL]}$. The indices of I, J denote the eleven-dimensional coordinates with $x^0, \cdots, x^9, x^{11}$. On the other hand, $S_{\rm YM}$ describes super Yang-Mills theory on two ten-dimensional orbifold planes denoted by $M^{10}_{(1)}$ and $M^{10}_{(2)}$ for $x^{11}=0$ and $x^{11}=\pi\rho$, where we choose x^{11} as the orbifold direction with the range $x^{11}\in[-\pi\rho,\pi\rho]$. It is given by[1]

$$\begin{aligned} S_{\rm YM} = &-\frac{1}{8\pi\kappa^2}\left(\frac{\kappa}{4\pi}\right)^{2/3}\int_{M^{10}_{(1)}} d^{10}x\,\sqrt{-g}\,{\rm tr}\left(F^{(1)}_{AB}F^{(1)AB}\right) \\ &-\frac{1}{8\pi\kappa^2}\left(\frac{\kappa}{4\pi}\right)^{2/3}\int_{M^{10}_{(2)}} d^{10}x\,\sqrt{-g}\,{\rm tr}\left(F^{(2)}_{AB}F^{(2)AB}\right) \\ &-\frac{1}{16\pi\kappa^2}\left(\frac{\kappa}{4\pi}\right)^{2/3}\int_{M^{10}_{(1)},\,M^{10}_{(2)}} d^{10}x\,\sqrt{-g}\left(R_{ABCD}R^{ABCD} - 4\,R_{AB}R^{AB} + R^2\right) \end{aligned} \tag{2.3}$$

where $F^{(1),(2)}_{AB}$ denote E_8 gauge fields living on the orbifold planes. The metric in $S_{\rm YM}$ is the ten-dimensional part g_{AB} of the eleven-dimensional metric with the ten-dimensional indices $A, B = 0, \cdots, 9$. The Gauss-Bonnet R^2 terms appear in the last line in (2.3).

If we consider M-theory on a smooth manifold, there is no anomaly. In contrast, on a S^1/Z_2 orbifold, there are gauge and gravitational anomalies owing to Z_2 singularities [17, 1, 2]. These anomalies can be canceled [1, 2] by the Green-Schwarz mechanism [18] with the E_8 gauge fields. At higher order in κ, additional interactions are required by the anomaly cancellation in terms of higher powers of F and R [1, 2, 9]. These interactions are investigated from a one-loop effect in type IIA string theory [19] or anomaly cancellation on world-volume of M5-brane in eleven dimensions [20, 21, 15, 16]. When we reduce this effective action to ten dimensions, the four-form field strength with the x^{11} index G_{ABC11} is promoted to a three-form field strength H_{ABC}. It includes the Yang-Mills and Lorentz Chern-Simons three-forms due to a modification of the Bianchi identity (2.10) by the Green-Schwarz mechanism [2, 9]. In addition, the Gauss-Bonnet R^2 terms are required by ten-dimensional supersymmetry pairing with the Lorentz Chern-Simons three-form [10]. In the following, we concentrate on the effective action up to order $\kappa^{2/3}$.

The Einstein equation and the Maxwell equation are given as

$$\begin{aligned} R_{IJ} - \frac{1}{2}g_{IJ}\,R = &-\frac{1}{24}\left(4\,G_{IKLM}G_J{}^{KLM} - \frac{1}{2}g_{IJ}\,G_{KLMN}G^{KLMN}\right) \\ &-\frac{1}{2\pi}\left(\frac{\kappa}{4\pi}\right)^{2/3}\left\{\delta(x^{11})\,T^{(1)}_{IJ} + \delta(x^{11}-\pi\rho)\,T^{(2)}_{IJ}\right\}\ , \end{aligned} \tag{2.4}$$

[1] In [15, 16], it is argued that we should multiply both $S_{\rm YM}$ and the Bianchi identity (2.10) by an additional factor of $2^{-1/3}$. But we take the original form of [2] for simplicity since this difference is not essential in the following discussion.

$$\partial_I \left(\sqrt{-g}\, G^{IJKL} \right) = \frac{\sqrt{2}}{1152}\, \epsilon^{JKLI_1 \cdots I_8}\, G_{I_1 \cdots I_4} G_{I_5 \cdots I_8} \tag{2.5}$$

where

$$T^{(i)}_{AB} = (g_{11,11})^{-1/2} \left\{ \mathrm{tr} \left(F^{(i)}_{AC} F^{(i)C}_{B} \right) - \frac{1}{4}\, g_{AB}\, \mathrm{tr} \left(F^{(i)}_{CD} F^{(i)CD} \right) + \frac{1}{4}\, \mathcal{G}_{AB} \right\} , \tag{2.6}$$

$$T^{(i)}_{11,11} = 0 \, . \tag{2.7}$$

We have defined $\mathcal{G}_{AB}$ in (2.6) as a variation of the Gauss-Bonnet terms with respect to the metric:

$$\delta \Big\{ \sqrt{-g}\, (R_{ABCD} R^{ABCD} - 4\, R_{AB} R^{AB} + R^2) \Big\} \equiv \sqrt{-g}\, \mathcal{G}_{AB}\, \delta g^{AB} \tag{2.8}$$

where

$$\begin{aligned} \mathcal{G}_{AB} = & -\frac{1}{2}\, g_{AB} (R_{CDEF} R^{CDEF} - 4\, R_{CD} R^{CD} + R^2) \\ & + 2\, R\, R_{AB} + 2\, R_{ACDE} R_B{}^{CDE} - 4\, R_{ACBD} R^{CD} - 4\, R_A{}^C R_{BC} \, . \end{aligned} \tag{2.9}$$

We note that the Einstein equation has source terms proportional to $\kappa^{2/3}$. The Bianchi identity also has source terms due to a modification [2] for anomaly cancellation and supersymmetry by the Green-Schwarz mechanism:

$$(dG)_{11ABCD} = -\frac{1}{2\sqrt{2}\, \pi} \left(\frac{\kappa}{4\pi} \right)^{2/3} \left\{ J^{(1)}_{ABCD}\, \delta(x^{11}) + J^{(2)}_{ABCD}\, \delta(x^{11} - \pi\rho) \right\} \tag{2.10}$$

where

$$J^{(i)}_{ABCD} = 6 \left\{ \mathrm{tr} \left(F^{(i)}_{[AB} F^{(i)}_{CD]} \right) - \frac{1}{2}\, \mathrm{tr} \left(R_{[AB} R_{CD]} \right) \right\} \;=\; \left(d\, \omega^{(i)}_3 \right)_{ABCD} , \tag{2.11}$$

and $\omega^{(i)}_3 \equiv \omega^{(i)\mathrm{YM}}_3 - \frac{1}{2}\, \omega^{\mathrm{L}}_3$ denote the Yang-Mills and Lorentz Chern-Simons three-forms. R in the trace is the curvature two-form and the trace is taken over the tangent space indices.

The above description of the Bianchi identity (2.10) is called the "*upstairs picture*" [2] in which we consider the x^{11} coordinate as a circle with two Z_2 singularities, namely ten-dimensional orbifold planes at $x^{11} = 0, \pi\rho$. On the other hand, there is another description called the "*downstairs picture*" in which the x^{11} coordinate is regarded as a segment with two boundaries at $x^{11} = 0, \pi\rho$. In this picture, the Bianchi identity is rewritten as

$$(dG)_{KLMNP} = 0 \tag{2.12}$$

with the boundary conditions

$$G_{ABCD} \Big|_{x^{11}=0} = -\frac{1}{4\sqrt{2}\, \pi} \left(\frac{\kappa}{4\pi} \right)^{2/3} J^{(1)}_{ABCD} , \tag{2.13}$$

$$G_{ABCD} \Big|_{x^{11}=\pi\rho} = \frac{1}{4\sqrt{2}\, \pi} \left(\frac{\kappa}{4\pi} \right)^{2/3} J^{(2)}_{ABCD} \tag{2.14}$$

where the additional factor of $1/2$ comes from changing a range of the x^{11} coordinate such that $\int_{-\pi\rho}^{\pi\rho} dx^{11} \to 2 \int_0^{\pi\rho} dx^{11}$.

As mentioned above, equations of motion and the Bianchi identity have source terms with $\delta(x^{11})$. This is the reason why the solution of the equations gets a non-trivial x^{11}-dependence and it is

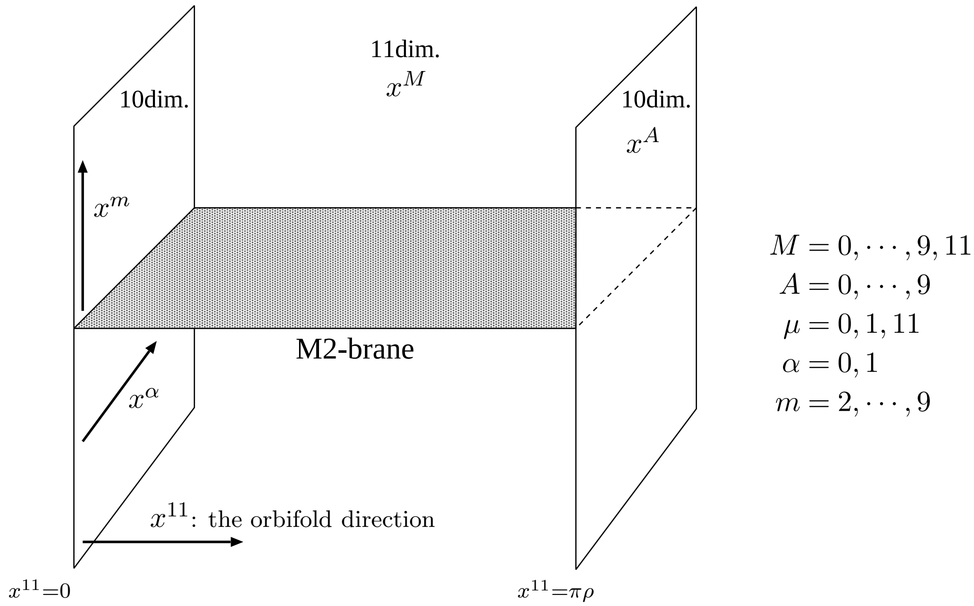

Figure 1: Indices of coordinates

regarded as the strong coupling effect from the point of view of ten-dimensional string theory. Neglecting the Gauss-Bonnet terms, the gauge 5-brane solution with the x^{11}-dependence was computed explicitly by Lalak et al.[11] In many cases, a contribution of the Gauss-Bonnet terms is neglected since it is higher order in derivatives. On the other hand, the $\mathrm{tr}R^2$ terms in the modified Bianchi identity (2.10) have played the significant role in this effective theory. For the Calabi-Yau compactification, for instance, the connection embedding [3, 4] with the $\mathrm{tr}R^2$ terms is very important. So, to discuss up to order $\kappa^{2/3}$, the Gauss-Bonnet terms should be considered like the $\mathrm{tr}R^2$ terms, because they are of the same order. The main aim of this paper is to investigate the effect of the Gauss-Bonnet terms in heterotic M-theory at low-energy.

3 The M2-brane solution

We consider the M2-brane solution up to order $\kappa^{2/3}$. There are two ways to embed this solution into heterotic M-theory depending on whether M2-brane is oriented orthogonal or parallel to the orbifold planes. It has been shown [11] that, neglecting the Gauss-Bonnet terms, the M2-brane ansatz makes the source terms vanish and we find the usual solution of eleven-dimensional supergravity [13]. In addition, for the Z_2 operation of the orbifold, the orthogonal M2-brane solution is a BPS solution preserving a quarter of the eleven-dimensional supersymmetry, while the parallel one does not preserve any supersymmetry. Then, the orthogonal M2-brane solution is expected to describe strongly coupled heterotic string effectively. On the other hand, if we take the the Gauss-Bonnet terms into account, they give a new contribution to source terms in the Einstein equation (2.4) with the non-trivial metric as we have seen in (2.6) and (2.9). In the following, we consider the usual orthogonal M2-brane ansatz but we will show that it receives a correction of order $\kappa^{2/3}$.

3.1 The ansatz and the linearization of equations

We split the eleven-dimensional coordinates x^M into x^μ $(\mu = 0, 1, 11)$ and x^m $(m = 2, 3, \cdots, 9)$ for the world-volume coordinates of M2-brane and the transverse coordinates respectively. Furthermore, we split x^μ into x^α $(\alpha = 0, 1)$ and x^{11} where x^α denotes the coordinates of projected M2-brane onto the orbifold planes, as summarized in Figure 1. We take an ansatz for the metric in the following form:

$$g_{MN} = \begin{pmatrix} g_{\alpha\beta} & & 0 \\ & g_{mn} & \\ 0 & & g_{11,11} \end{pmatrix} = \begin{pmatrix} e^{2A}\,\eta_{\alpha\beta} & & 0 \\ & e^{2B}\,\delta_{mn} & \\ 0 & & e^{2X} \end{pmatrix} \tag{3.1}$$

with

$$\begin{aligned} A &= \frac{1}{3}\,Y_0 + \kappa^{2/3}\Phi_{1A} + \mathrm{O}(\kappa^{4/3})\ , \\ B &= -\frac{1}{6}\,Y_0 + \kappa^{2/3}\Phi_{1B} + \mathrm{O}(\kappa^{4/3})\ , \\ X &= \frac{1}{3}\,Y_0 + \kappa^{2/3}\Phi_{1X} + \mathrm{O}(\kappa^{4/3}) \end{aligned} \tag{3.2}$$

where the subscripts " 0 " and " 1 " denote zero-th and first order in $\kappa^{2/3}$ respectively. The zero-th order term Y_0 is a known function defined later in (3.10). At order κ^0, this metric describes orthogonal M2-brane which spreads on the directions of x^0, x^1 and x^{11} at $x^m = 0$. For the three-form C_{MNL}, we require $C_{\mu\nu\rho} \equiv \frac{1}{\sqrt{2}\,3!}\,\epsilon_{\mu\nu\rho}\,e^C$ and set others to zero with the field strength:

$$G_{m\mu\nu\rho} = \frac{1}{\sqrt{2}}\,\epsilon_{\mu\nu\rho}\,\partial_m\,e^C\ , \quad \text{others} = 0 \tag{3.3}$$

where

$$C = Y_0 + \kappa^{2/3}\Phi_{1C} + \mathrm{O}(\kappa^{4/3})\ . \tag{3.4}$$

We note that, at order κ^0, the ansatz leads to the usual M2-brane solution [13] with $A = X = 1/3\,C$ and $B = -1/6\,C$. At order $\kappa^{2/3}$, fields depend on x^{11} as well as r due to the source terms with $\delta(x^{11})$ in equations of motion (2.4). Then $\Phi_{1A}, \Phi_{1B}, \Phi_{1X}$ and Φ_{1C} are functions of x^{11} and r. We turn off the E_8 gauge fields to see the effect of the pure Gauss-Bonnet terms on the M2-brane solution.

Now, we linearize equations of motion by the ansatz, (3.1) and (3.3). Substituting the ansatz to the Einstein equation (2.4), we find the following conditions for $(\alpha, \beta), (m, n)$ and $(11, 11)$ components of indices:

$$\begin{aligned} (\alpha,\beta)\ : \qquad & \frac{1}{2}\,e^{Y_0}\Box e^{-Y_0} \\ & +\kappa^{2/3}\Big[\partial Y_0\,\partial Y_0(-\Phi_{1A} - \frac{1}{2}\,\Phi_{1X} + \frac{1}{2}\,\Phi_{1C}) + \partial Y_0(-3\partial\Phi_{1B} + \frac{1}{2}\,\partial\Phi_{1C}) \\ & \qquad +\Box\Phi_{1A} + 7\Box\Phi_{1B} + \Box\Phi_{1X} + e^{-Y_0}\,\partial_{11}^2(\phi_A + 8\phi_B)\Big] \\ & = \frac{1}{2}\,\kappa^{2/3}\Big\{\delta(x^{11}) + \delta(x^{11} - \pi\rho)\Big\}\,J_1 + \mathrm{O}(\kappa^{4/3})\ , \end{aligned} \tag{3.5}$$

$$(m,n)\ : \qquad \kappa^{2/3}\Big[(\partial Y_0\,\partial Y_0 + \partial Y_0\,\partial)(6\Phi_{1A} + 3\Phi_{1X} - 3\Phi_{1C})$$

$$+14\Box\Phi_{1A}+42\Box\Phi_{1B}+7\Box\Phi_{1X}+e^{-Y_0}\,\partial_{11}^2(16\phi_A+56\phi_B)\Big]$$
$$=\kappa^{2/3}\Big\{\delta(x^{11})+\delta(x^{11}-\pi\rho)\Big\}\,J_2+\mathrm{O}(\kappa^{4/3})\ , \tag{3.6}$$

$$(11,11)\ :\qquad \frac{1}{2}\,e^{Y_0}\Box e^{-Y_0}$$
$$+\kappa^{2/3}\Big[\partial Y_0\,\partial Y_0(-\Phi_{1A}-\frac{1}{2}\,\Phi_{1X}+\frac{1}{2}\,\Phi_{1C})+\partial Y_0(-3\partial\Phi_{1B}+\frac{1}{2}\,\partial\Phi_{1C})$$
$$+2\Box\Phi_{1A}+7\Box\Phi_{1B}\Big]+\mathrm{O}(\kappa^{4/3})=0 \tag{3.7}$$

where ∂ denotes the derivative with respect to x^m and $\Box$ denotes $\delta^{mn}\partial_m\partial_n$. We define J_1 and J_2 as contributions of $T^{(i)}$ in (2.6) to the source terms. Since we turn off E_8 gauge fields $F^{(i)}$, only the variation $\mathcal{G}$ of the Gauss-Bonnet terms in (2.9) appears in J_1 and J_2. The Maxwell equation (2.5) leads to two conditions:

$(\nabla_m G^{\mu\nu\rho m})$:

$$-\,\Box e^{-Y_0}+\kappa^{2/3}\,e^{-Y_0}\,\{\partial Y_0(-2\partial\Phi_{1A}+6\partial\Phi_{1B}-\partial\Phi_{1X})+\Box\Phi_{1C}\}+\mathrm{O}(\kappa^{4/3})=0\ , \tag{3.8}$$

$(\nabla_\mu G^{\mu\nu\rho m})$:

$$\kappa^{2/3}\,e^{-Y_0}\{\partial Y_0(-2\partial_{11}\Phi_{1A}+6\partial_{11}\Phi_{1B}-\partial_{11}\Phi_{1X}+\partial_{11}\Phi_{1C})+\partial_{11}\partial\Phi_{1C}\}+\mathrm{O}(\kappa^{4/3})=0\ . \tag{3.9}$$

We consider the Bianchi identity (2.10). Since we turn off the gauge fields, source terms are given by $\mathrm{tr}R^2$ terms. However it is straightforward to show [11] that the $\mathrm{tr}R^2$ vanishes at order κ^0 for its anti-symmetric indices upon substituting the ansatz of the metric (3.1). For this reason, the Bianchi identity does not lead to any additional condition up to order $\kappa^{2/3}$.

At order κ^0, the above equations of motion (3.5)–(3.9) lead to the usual field equation $\Box e^{-Y_0}=0$ for eleven-dimensional supergravity. The solution is given by

$$e^{-Y_0}=1+\frac{Q}{r^6} \tag{3.10}$$

where Q denotes a charge of M2-brane [13].

Next, we compute the source terms, J_1 and J_2, in the Einstein equation, (3.5) and (3.6). Substituting the metric (3.1) to $\mathcal{G}$ in (2.9), we get

$$g^{\alpha\beta}\,\mathcal{G}_{\alpha\beta}=e^{\frac{2}{3}Y_0}\Big(-\frac{2}{3}\,\Box Y_0\,\Box Y_0+\frac{2}{3}\,\partial_m\partial_n Y_0\,\partial_i\partial_j Y_0\,\delta^{mi}\delta^{nj}+\frac{1}{3}\,\Box Y_0\,\partial_m Y_0\,\partial_n Y_0\,\delta^{mn}$$
$$-\frac{10}{9}\,\partial_m\partial_n Y_0\,\partial_i Y_0\,\partial_j Y_0\,\delta^{mi}\delta^{nj}-\frac{7}{54}\,\partial_m Y_0\,\partial_n Y_0\,\partial_i Y_0\,\partial_j Y_0\,\delta^{mn}\delta^{ij}\Big)+\mathrm{O}(\kappa^{2/3})$$
$$=36Q^2(r^6+Q)^{-\frac{14}{3}}\Big(-\frac{14}{3}\,Q^2+\frac{196}{3}\,Qr^6+\frac{112}{3}\,r^{12}\Big)\ +\ \mathrm{O}(\kappa^{2/3})\ , \tag{3.11}$$

$$g^{mn}\,\mathcal{G}_{mn}=e^{\frac{2}{3}Y_0}\Big(4\,\Box Y_0\,\Box Y_0-4\,\partial_k\partial_l Y_0\,\partial_i\partial_j Y_0\,\delta^{ki}\delta^{lj}-\frac{8}{9}\,\Box Y_0\,\partial_i Y_0\,\partial_j Y_0\,\delta^{ij}$$
$$-\frac{16}{3}\,\partial_k\partial_l Y_0\,\partial_i Y_0\,\partial_j Y_0\,\delta^{ki}\delta^{lj}-\frac{7}{27}\,\partial_k Y_0\,\partial_l Y_0\,\partial_i Y_0\,\partial_j Y_0\,\delta^{kl}\delta^{ij}\Big)+\mathrm{O}(\kappa^{2/3})$$
$$=36Q^2(r^6+Q)^{-\frac{14}{3}}\Big(\frac{308}{3}\,Q^2+112\,Qr^6-224\,r^{12}\Big)\ +\ \mathrm{O}(\kappa^{2/3})\ . \tag{3.12}$$

From (2.4), (3.11) and (3.12), we find

$$J_1 = 36\alpha\,(r^6+Q)^{-4}\,r^{-4}Q^2\Big(-\frac{14}{3}\,Q^2+\frac{196}{3}\,Qr^6+\frac{112}{3}\,r^{12}\Big)\,, \tag{3.13}$$

$$J_2 = 36\alpha\,(r^6+Q)^{-4}\,r^{-4}Q^2\Big(\frac{308}{3}\,Q^2+112\,Qr^6-224\,r^{12}\Big) \tag{3.14}$$

where $\alpha \equiv 1/8\pi(4\pi)^{2/3}$.

We obtained equations for $\Phi_{1A}, \Phi_{1B}, \Phi_{1X}$ and Φ_{1C} at order $\kappa^{2/3}$. At this stage, these four functions Φ_1 are unknown, while we have two conditions from the Maxwell equation, (3.8) and (3.9), and three conditions from the Einstein equation (3.5)–(3.7). One may wonder there are too many conditions. But the two conditions from the Maxwell equation are related by an arbitrary function f, as we will see below. Then the number of functions matches to the number of independent conditions and we can solve these equations.

We separate Φ_1 into x^{11}-independent and x^{11}-dependent parts

$$\Phi_{1A} \equiv A_1 + \phi_A\ , \quad \Phi_{1B} \equiv B_1 + \phi_B\ , \quad \Phi_{1X} \equiv X_1 + \phi_X\ , \quad \Phi_{1C} \equiv C_1 + \phi_C \tag{3.15}$$

where A_1, B_1, X_1 and C_1 depend only on r, while ϕ_A, ϕ_B, ϕ_X and ϕ_C are functions of x^{11} and r. To determine these decompositions uniquely, we demand [11]

$$\int_0^{\pi\rho} \phi_A\, dx^{11} = \int_0^{\pi\rho} \phi_B\, dx^{11} = \int_0^{\pi\rho} \phi_X\, dx^{11} = 0\ . \tag{3.16}$$

For ϕ_C, on the other hand, we need a different condition. If we require the same condition $\int_0^{\pi\rho} \phi_C\, dx^{11} = 0$ as other ϕ, we obtain two independent conditions for ϕ_A and ϕ_B from the (α, β) and (m, n) components of the Einstein equation, (3.5) and (3.6). Together with the $(11, 11)$ component (3.7), we find three conditions which are too strong for ϕ_A and ϕ_B. So, there is no solution with $\int_0^{\pi\rho} \phi_C\, dx^{11} = 0$. For this reason, we require $\int_0^{\pi\rho} \phi_C\, dx^{11} \neq 0$ and will determine the value by equations of motion, below.

Now, let us decompose the equations of motion (3.5)–(3.9) according to (3.15) and (3.16). For the Maxwell equation, (3.8) and (3.9), we find that

$$\partial Y_0(-2\partial A_1 + 6\partial B_1 - \partial X_1) + \Box C_1 + \partial Y_0(-2\partial\phi_A + 6\partial\phi_B - \partial\phi_X) + \Box\phi_C = 0\ , \tag{3.17}$$

$$\partial Y_0(-2\partial_{11}\phi_A + 6\partial_{11}\phi_B - \partial_{11}\phi_X + \partial_{11}\phi_C) + \partial_{11}\partial\phi_C = 0\ . \tag{3.18}$$

For the x^{11}-dependent part, (3.18) is integrated over x^{11} with a result

$$\partial Y_0(-2\phi_A + 6\phi_B - \phi_X + \phi_C) + \partial\phi_C + \partial Y_0\, f = 0 \tag{3.19}$$

where f is an arbitrary function of r arising from the integration. Differentiating (3.19) by x^m, we have

$$\partial Y_0(-2\partial\phi_A + 6\partial\phi_B - \partial\phi_X) + \Box\phi_C + \partial Y_0\, \partial f = 0\ . \tag{3.20}$$

Substituting (3.20) to (3.17) yields, for the x^{11}-independent part,

$$\partial Y_0(-2\partial A_1 + 6\partial B_1 - \partial X_1) + \Box C_1 - \partial Y_0\, \partial f = 0 \tag{3.21}$$

and, after integrating over x^m, it leads to

$$\partial Y_0(-2A_1 + 6B_1 - X_1 + C_1) + \partial C_1 - \partial Y_0\, f + g = 0 \tag{3.22}$$

where g is a constant of integration.

Next, we turn to the Einstein equation. For the $(11, 11)$ component, we substitute (3.19) to (3.7) and integrate it over x^{11}. Then, according to (3.16), we find that

$$-3\partial Y_0 \partial Y_0\, B_1 - 3\partial Y_0 \partial B_1 + 2\Box A_1 + 7\Box B_1 - \frac{1}{2}\, g = 0\ , \tag{3.23}$$

$$-3\partial Y_0 \partial Y_0\, \phi_B - 3\partial Y_0 \partial \phi_B + 2\Box \phi_A + 7\Box \phi_B = 0\ . \tag{3.24}$$

Consider the (α, β) component. Substituting (3.7) to (3.5), we obtain

$$-\Box\Phi_A + \Box\Phi_X + e^{-Y_0}\, \partial_{11}^2(\phi_A + 8\phi_B) = \frac{1}{2}\left\{\delta(x^{11}) + \delta(x^{11} - \pi\rho)\right\} J_1\ . \tag{3.25}$$

In the "*downstairs picture*", this equation is rewritten as

$$-\Box\Phi_A + \Box\Phi_X + e^{-Y_0}\, \partial_{11}^2(\phi_A + 8\phi_B) = 0 \tag{3.26}$$

with boundary conditions

$$[e^{-Y_0}\, \partial_{11}(\phi_A + 8\phi_B)]\Big|_{x^{11}=0} = \frac{1}{4}\, J_1\ , \tag{3.27}$$

$$[e^{-Y_0}\, \partial_{11}(\phi_A + 8\phi_B)]\Big|_{x^{11}=\pi\rho} = -\frac{1}{4}\, J_1\ . \tag{3.28}$$

Integrating (3.26) over x^{11}, we find

$$-\Box A_1 + \Box X_1 = \frac{1}{2}\, J_1\, \frac{1}{\pi\rho} \tag{3.29}$$

for the x^{11}-independent part, and

$$\begin{aligned} -\Box\phi_A + \Box\phi_X + e^{-Y_0}\, \partial_{11}^2(\phi_A + 8\phi_B) &= -\frac{1}{2}\, J_1\, \frac{1}{\pi\rho}\ ,\\ [e^{-Y_0}\, \partial_{11}(\phi_A + 8\phi_B)]\Big|_{x^{11}=0} &= \frac{1}{4}\, J_1\ ,\\ [e^{-Y_0}\, \partial_{11}(\phi_A + 8\phi_B)]\Big|_{x^{11}=\pi\rho} &= -\frac{1}{4}\, J_1 \end{aligned} \tag{3.30}$$

for the x^{11}-dependent part. The remaining (m, n) component is also worked out in the same way as the (α, β) component. In the *downstairs picture*, results are

$$(\partial Y_0\, \partial Y_0 + \partial Y_0\, \partial)(6A_1 + 3X_1 - 3C_1) + 14\Box A_1 + 42\Box B_1 + 7\Box X_1 = J_2\, \frac{1}{\pi\rho} \tag{3.31}$$

and

$$\begin{aligned} &(\partial Y_0\, \partial Y_0 + \partial Y_0\, \partial)(6\phi_A + 3\phi_X - 3\phi_C) + 14\Box\phi_A + 42\Box\phi_B + 7\Box\phi_X\\ &\quad + e^{-Y_0}\, \partial_{11}^2(16\phi_A + 56\phi_B) = -J_2\, \frac{1}{\pi\rho}\ ,\\ &\left[-3\partial Y_0\, \partial \int \phi_C\, dx^{11} + e^{-Y_0}\, \partial_{11}(16\phi_A + 56\phi_B)\right]\Big|_{x^{11}=0} = \frac{1}{2}\, J_2\ ,\\ &\left[-3\partial Y_0\, \partial \int \phi_C\, dx^{11} + e^{-Y_0}\, \partial_{11}(16\phi_A + 56\phi_B)\right]\Big|_{x^{11}=\pi\rho} = -\frac{1}{2}\, J_2\ . \end{aligned} \tag{3.32}$$

We have finished decomposing all equations of motion, which are summarized in the table below.

	The x^{11}-independent part	The x^{11}-dependent part
The Maxwell equation	(3.21)	(3.20)
The Einstein equation		
$(\alpha\beta)$	(3.29)	(3.30)
(m,n)	(3.31)	(3.32)
$(11,11)$	(3.23)	(3.24)

3.2 Solving equations at order $\kappa^{2/3}$

We solve the equations in two asymptotic regions, $r^6 \gg Q$ and $r^6 \ll Q$, by expanding fields in r. In these regions we find asymptotically, in $r^6 \gg Q$:

$$e^{-Y_0} = 1\ , \quad \partial^m Y_0 = 6\,Q\,r^{-8}x^m\ , \quad J_1 = j_1\,\alpha Q^2\,r^{-16}\ , \quad J_2 = j_2\,\alpha Q^2\,r^{-16} \tag{3.33}$$

where $j_1 \equiv 1344$ and $j_2 \equiv -8064$, while in $r^6 \ll Q$:

$$e^{-Y_0} = r^{-6}Q\ , \quad \partial^m Y_0 = 6\,r^{-2}x^m\ , \quad J_1 = \tilde{j}_1\,\alpha\,r^{-4}\ , \quad J_2 = \tilde{j}_2\,\alpha\,r^{-4} \tag{3.34}$$

where $\tilde{j}_1 \equiv -168$ and $\tilde{j}_2 \equiv 3696$.

3.2.1 $r^6 \gg Q$ region

First, we consider the x^{11}-dependent part. Since source terms become zero at $r \to \infty$ limit by (3.33), we are interested in a solution which is regular at $r \to \infty$ limit. We should expand fields in r^{-1} as

$$\phi(r, x^{11}) = \sum_{i=0}^{\infty} \psi_{p+i}\,r^{-(p+i)} \tag{3.35}$$

for a certain integer p where ψ_i is a function of x^{11}. In this expression, $\psi_p\,r^{-p}$ is dominant and we can regard $\psi_i\,r^{-i}$ for $i > p$ as a small correction. If we assume $\phi \sim r^{-p}$ approximately without the x^{11}-dependence, then p can be determined by source terms, J_1 and J_2, as follows.

1. From the $(11,11)$ component of the Einstein equation (3.24), we find $\phi_A \sim \phi_B$.
2. The (α,β) component (3.30) yields $\phi_A \sim \phi_B \sim J_1 \sim r^{-16}$.
3. From boundary conditions of the (m,n) component (3.32), we obtain
$$\partial Y_0\,\partial \int \phi_C\,dx^{11}\Big|_{x^{11}=0,\pi\rho} \sim r^{-16}$$
which tells us $\phi_C \sim r^{-8}$.
4. We integrate the Maxwell equation (3.20) over x^{11} from 0 to $\pi\rho$ with a result
$$\Box \int_0^{\pi\rho} \phi_C\,dx^{11} + \partial Y_0\,\partial f\,\pi\rho = 0$$
which amounts to $f \sim r^{-2}$. Moreover, substituting the result to (3.20) again, we obtain $\phi_X \sim \phi_A \sim \phi_B \sim r^{-16}$.

The above discussion is very rough and ignores the x^{11}-dependence, but we find a significant result

$$\phi_A = \sum_{i=0}^{\infty}\psi_{16+i}^{(A)}\,r^{-16-i}\ , \quad \phi_B = \sum_{i=0}^{\infty}\psi_{16+i}^{(B)}\,r^{-16-i}\ , \quad \phi_X = \sum_{i=0}^{\infty}\psi_{16+i}^{(X)}\,r^{-16-i}\ ,$$
$$\phi_C = \sum_{i=0}^{\infty}\psi_{8+i}^{(C)}\,r^{-8-i}\ , \quad f = \sum_{i=0}^{\infty}P_{2+i}^{(f)}\,r^{-2-i} \tag{3.36}$$

where $\psi_i^{(A)}, \psi_i^{(B)}, \psi_i^{(X)}$ and $\psi_i^{(C)}$ are functions of x^{11}, while $P_i^{(f)}$ is a constant. We note that $\int_0^{\pi\rho} \phi_C \, dx^{11}$ is determined by f as mentioned above.

Next, we expand the Maxwell equation (3.20) and the Einstein equation, (3.24), (3.30) and (3.32), in r. According to (3.36), it is sufficient to consider the following parts of equations.

The Maxwell equation:

$$\sum_i \Big[-6(i-8)Q\Big\{ -2\psi_{i-8}^{(A)} + 6\psi_{i-8}^{(B)} - \psi_{i-8}^{(X)} \Big\} \\ +(i-2)(i-8)\psi_{i-2}^{(C)} - 6(i-8)QP_{i-8}^{(f)} \Big] r^{-i} = 0 \,, \qquad \underline{i \geq 10} \tag{3.37}$$

The Einstein equation

$$(\alpha,\beta) : \quad \sum_i \Big[(i-2)(i-8)\Big\{ -\psi_{i-2}^{(A)} + \psi_{i-2}^{(X)} \Big\} + \partial_{11}^2 \psi_i^{(A)} + 8\partial_{11}^2 \psi_i^{(B)} \Big] r^{-i} \\ = -\frac{1}{2}\, j_1\, \alpha Q^2\, r^{-16}\, \frac{1}{\pi\rho} \,, \qquad \underline{i \geq 16} \tag{3.38}$$

$$(m,n) : \quad \sum_i \Big[36Q^2 \Big\{ 6\psi_{i-14}^{(A)} + 3\psi_{i-14}^{(X)} - 3\psi_{i-14}^{(C)} \Big\} \\ -6(i-8)Q\Big\{ 6\psi_{i-8}^{(A)} + 3\psi_{i-8}^{(X)} - 3\psi_{i-8}^{(C)} \Big\} \\ +(i-2)(i-8)\Big\{ 14\psi_{i-2}^{(A)} + 42\psi_{i-2}^{(B)} + 7\psi_{i-2}^{(X)} \Big\} \\ +16\partial_{11}^2\psi_i^{(A)} + 56\partial_{11}^2\psi_i^{(B)} \Big] r^{-i} = -j_2\, \alpha Q^2\, r^{-16}\, \frac{1}{\pi\rho} \,, \qquad \underline{i \geq 16} \tag{3.39}$$

$$(11,11) : \quad \sum_i \Big[-108Q^2\psi_{i-14}^{(B)} + 18(i-8)Q\psi_{i-8}^{(B)} \\ +(i-2)(i-8)\Big\{ 2\psi_{i-2}^{(A)} + 7\psi_{i-2}^{(B)} \Big\} \Big] r^{-i} = 0 \,, \qquad \underline{i \geq 18} \tag{3.40}$$

We note that it is always necessary to keep boundary conditions in mind with the (α, β) and (m, n) components.

Let us determine ψ_i and $P_i^{(f)}$. In $r^6 \gg Q$ region, it is enough to consider the dominant part: $\psi_{16}^{(A)}$, $\psi_{16}^{(B)}$, $\psi_{16}^{(X)}$, $\psi_8^{(C)}$ and $P_2^{(f)}$. $i = 18$ part of (3.40) gives $\psi_{16}^{(A)} = -7/2\, \psi_{16}^{(B)}$, and then $i = 16$ part of (3.38) yields

$$\partial_{11}^2 \psi_{16}^{(B)} = -\frac{1}{9}\, j_1\, \alpha Q^2\, \frac{1}{\pi\rho} \tag{3.41}$$

with boundary conditions

$$\partial_{11}\psi_{16}^{(B)}\Big|_{x^{11}=0} = \frac{1}{18}\, j_1\, \alpha Q^2 \,, \quad \partial_{11}\psi_{16}^{(B)}\Big|_{x^{11}=\pi\rho} = -\frac{1}{18}\, j_1\, \alpha Q^2 \,. \tag{3.42}$$

From these conditions, we have

$$\psi_{16}^{(A)} = -\frac{7}{2}\, \psi_{16}^{(B)} = \frac{1568}{3}\, \alpha Q^2 \left\{ \frac{1}{2\pi\rho}(x^{11})^2 - \frac{1}{2}\, x^{11} + \frac{\pi\rho}{12} \right\} \,. \tag{3.43}$$

From $i = 16$ part of (3.39) and $i = 10$ part of (3.37), $\psi_8^{(C)}$ and $P_2^{(f)}$ are given by

$$\psi_8^{(C)} = 56\, \alpha Q\, \frac{1}{\pi\rho} \,, \tag{3.44}$$

$$P_2^{(f)} = \frac{224}{3}\, \alpha\, \frac{1}{\pi\rho} \,. \tag{3.45}$$

We note that the x^{11}-dependence of $\psi^{(A)}_{16}$ and $\psi^{(B)}_{16}$ agrees with the result of the gauge 5-brane[2]. We consider $\psi^{(X)}_{16}$ which remains unknown. Since $i=24$ part of (3.37) relates $\psi^{(X)}_{16}$, $\psi^{(C)}_{22}$ and $P^{(f)}_{16}$, it is necessary to use $i=18$ part of (3.39) and $i=20$ part of (3.37). Solving these conditions, we find that

$$\psi^{(X)}_{16} = \psi^{(B)}_{16} = -\frac{448}{3}\,\alpha Q^2 \left\{ \frac{1}{2\pi\rho}(x^{11})^2 - \frac{1}{2}\,x^{11} + \frac{\pi\rho}{12} \right\} , \tag{3.46}$$

$$\psi^{(C)}_{10} = 0 , \tag{3.47}$$

$$11\psi^{(C)}_{22} - 3QP^{(f)}_{16} = 36Q\psi^{(B)}_{16} = -5376\,\alpha Q^3 \left\{ \frac{1}{2\pi\rho}(x^{11})^2 - \frac{1}{2}\,x^{11} + \frac{\pi\rho}{12} \right\} \tag{3.48}$$

where $P^{(f)}_{16}$ is determined by $i=30$ part of (3.39). $i=12$ part of (3.37) leads to $P^{(f)}_{4} = 0$.

Next, we turn to the x^{11}-independent part. By the same discussion on the x^{11}-dependent part, (3.29), (3.31) and (3.23) yield $\Box A \sim \Box B \sim \Box X \sim r^{-16}$ which leads to $A \sim B \sim X \sim r^{-14}$ and $g=0$. (3.21) gives $\Box C \sim \partial Y_0\, \partial f$, namely $C \sim r^{-8}$. Consequently, we find expansions of fields in r such that

$$A_1 = \sum_{i=0}^{\infty} P^{(A)}_{14+i}\, r^{-14-i} , \quad B_1 = \sum_{i=0}^{\infty} P^{(B)}_{14+i}\, r^{-14-i} , \quad X_1 = \sum_{i=0}^{\infty} P^{(X)}_{14+i}\, r^{-14-i} ,$$
$$C_1 = \sum_{i=0}^{\infty} P^{(C)}_{8+i}\, r^{-8-i} \tag{3.49}$$

where $P^{(A)}_i, P^{(B)}_i, P^{(X)}_i$ and $P^{(C)}_i$ are constants. For expansions of equations of motion (3.21), (3.23), (3.29) and (3.31), we only need the following parts of equations.

The Maxwell equation:

$$\sum_i \Big[-6(i-8)Q\Big\{ -2P^{(A)}_{i-8} + 6P^{(B)}_{i-8} - P^{(X)}_{i-8} \Big\} + (i-2)(i-8)P^{(C)}_{i-2}$$
$$+6(i-8)QP^{(f)}_{i-8} \Big]\, r^{-i} = 0 , \qquad \underline{i \geq 10} \tag{3.50}$$

The Einstein equation

$$(\alpha,\beta) : \quad \sum_i (i-2)(i-8)\Big\{ -P^{(A)}_{i-2} + P^{(X)}_{i-2} \Big\}\, r^{-i} = \frac{1}{2}\, j_1\, \alpha Q^2\, r^{-16}\, \frac{1}{\pi\rho} , \qquad \underline{i \geq 16} \tag{3.51}$$

$$(m,n) : \quad \sum_i \Big[36Q^2 \Big\{ 6P^{(A)}_{i-14} + 3P^{(X)}_{i-14} - 3P^{(C)}_{i-14} \Big\}$$
$$-6(i-8)Q\Big\{ 6P^{(A)}_{i-8} + 3P^{(X)}_{i-8} - 3P^{(C)}_{i-8} \Big\}$$
$$+(i-2)(i-8)\Big\{ 14P^{(A)}_{i-2} + 42P^{(B)}_{i-2} + 7P^{(X)}_{i-2} \Big\} \Big]\, r^{-i}$$
$$= j_2\, \alpha Q^2\, r^{-16}\, \frac{1}{\pi\rho} , \qquad \underline{i \geq 16} \tag{3.52}$$

$$(11,11) : \quad \sum_i \Big[-3\cdot 36Q^2 P^{(B)}_{i-14} + 18(i-8)QP^{(B)}_{i-8}$$
$$+(i-2)(i-8)\Big\{ 2P^{(A)}_{i-2} + 7P^{(B)}_{i-2} \Big\} \Big]\, r^{-i} = 0 , \qquad \underline{i \geq 16} \tag{3.53}$$

[2] The x^{11}-dependence of our result coincides with that of the gauge 5-brane [11] when we set $\sigma_1 = \sigma_2$ in the notation of [11]. These σ_1 and σ_2 denote sizes of two SU(2) instantons on orbifold planes at $x^{11} = 0, \pi\rho$ and specify the two gauge fields independently as $\mathrm{tr}{F^{(i)}}^2 \sim \Box^2 \ln(1 + {\sigma_i}^2/r^2)$.

From (3.50), (3.51) and (3.53), we find three conditions

$$P_8^{(C)} = -\frac{3}{4}\,QP_2^{(f)}\ , \quad P_{14}^{(X)} = P_{14}^{(A)} + \frac{1}{224}\,j_1\,\alpha\,Q^2\,\frac{1}{\pi\rho}\ , \quad P_{14}^{(B)} = -\frac{2}{7}\,P_{14}^{(A)}\ . \tag{3.54}$$

Recall that $P_2^{(f)}$ is given by (3.45). Substituting these conditions to (3.52), we have

$$P_{14}^{(A)} = -\frac{7}{2}\,P_{14}^{(B)} = -\frac{14}{3}\alpha Q^2\,\frac{1}{\pi\rho}\ , \tag{3.55}$$

$$P_{14}^{(X)} = \frac{4}{3}\alpha Q^2\,\frac{1}{\pi\rho}\ , \tag{3.56}$$

$$P_8^{(C)} = -56\,\alpha Q\,\frac{1}{\pi\rho}\ . \tag{3.57}$$

So far, we have determined ψ_i and P_i asymptotically. From the results (3.43)–(3.46) and (3.55)–(3.57), we obtain the solution of field equations as the dominant forms:

$$\begin{aligned}
&-\frac{2}{7}\,A_1 = B_1 = X_1 = \frac{4}{3}\,\alpha Q^2\,r^{-14}\,\frac{1}{\pi\rho}\ , \\
&\quad C_1 = -56\,\alpha Q\,r^{-8}\,\frac{1}{\pi\rho}\ , \qquad\qquad f = \frac{224}{3}\,\alpha\,r^{-2}\,\frac{1}{\pi\rho}\ , \\
&-\frac{2}{7}\,\phi_A = \phi_B = \phi_X = -\frac{448}{3}\,\alpha Q^2\,r^{-16}\left\{\frac{1}{2\pi\rho}(x^{11})^2 - \frac{1}{2}\,x^{11} + \frac{\pi\rho}{12}\right\}\ , \\
&\quad \phi_C = 56\,\alpha Q\,r^{-8}\,\frac{1}{\pi\rho}\ .
\end{aligned} \tag{3.58}$$

We note that $\psi_8^{(C)}$ and $P_8^{(C)}$ are identical but opposite in sign. So, Φ_{1C} vanishes at this order. Moreover, $\psi_8^{(C)}$ is independent of x^{11} accidentally, but $\psi_i^{(C)}$ for higher i depends on x^{11} such as $\psi_{22}^{(C)}$ in (3.48).

3.2.2 $r^6 \ll Q$ region

We first consider the x^{11}-dependent part, and repeat the procedure similar to the one in $r^6 \gg Q$ region. From (3.34), source terms are $J_1 \sim J_2 \sim r^{-4}$. Suppose $\phi \sim r^p$ approximately, except for the x^{11}-dependence. Then equations of motion lead to $\phi_A \sim \phi_B \sim \phi_X \sim r^2$ and $\phi_C \sim f \sim r^{-2}$. So, we find expansions of fields as

$$\begin{aligned}
\phi_A = \sum_{i=0}^{\infty}\psi_{2+i}^{(A)}\,r^{2+i}\ , \quad \phi_B = \sum_{i=0}^{\infty}\psi_{2+1}^{(B)}\,r^{2+i}\ , \quad \phi_X = \sum_{i=0}^{\infty}\psi_{2+i}^{(X)}\,r^{2+i}\ , \\
\phi_C = \sum_{i=0}^{\infty}\psi_{-2+i}^{(C)}\,r^{-2+i}\ , \quad f = \sum_{i=0}^{\infty}P_{-2+i}^{(f)}\,r^{-2+i}
\end{aligned} \tag{3.59}$$

where $\psi_i^{(A)}, \psi_i^{(B)}, \psi_i^{(X)}$ and $\psi_i^{(C)}$ are functions of x^{11}, while $P_i^{(f)}$ is a constant. Expanding equations of motion by (3.59), we find the following results.

The Maxwell equation:

$$\sum_i\ \left[6i\Big\{-2\psi_i^{(A)} + 6\psi_i^{(B)} - \psi_i^{(X)}\Big\} + i(i+6)\psi_i^{(C)} + 6iP_i^{(f)}\right] r^{i-2} = 0\ , \qquad \underline{i \geq -2} \tag{3.60}$$

The Einstein equation

$$(\alpha,\beta):\quad \sum_i\Big[i(i+6)\Big\{-\psi_i^{(A)}+\psi_i^{(X)}\Big\}+Q\Big\{\partial_{11}^2\psi_{i+4}^{(A)}+8\partial_{11}^2\psi_{i+4}^{(B)}\Big\}\Big]\,r^{i-2}$$
$$=-\frac{1}{2}\,\tilde{j}_1\,\alpha\,r^{-4}\,\frac{1}{\pi\rho}\,,\qquad \underline{i\geq -2} \tag{3.61}$$
$$(m,n):\quad \sum_i\Big[(36+6i)\Big\{6\psi_i^{(A)}+3\psi_i^{(X)}-3\psi_i^{(C)}\Big\}+i(i+6)\Big\{14\psi_i^{(A)}+42\psi_i^{(B)}+7\psi_i^{(X)}\Big\}$$
$$+Q\Big\{16\partial_{11}^2\psi_{i+4}^{(A)}+56\partial_{11}^2\psi_{i+4}^{(B)}\Big\}\Big]\,r^{i-2}=-\tilde{j}_2\,\alpha\,r^{-4}\,\frac{1}{\pi\rho}\,,\qquad \underline{i\geq -2} \tag{3.62}$$
$$(11,11):\quad \sum_i\,(i+6)\Big\{2i\psi_i^{(A)}+(7i-18)\psi_i^{(B)}\Big\}\,r^{i-2}=0\,,\qquad \underline{i\geq 2} \tag{3.63}$$

Solving these equations, we obtain

$$\psi_2^{(A)}=\psi_2^{(B)}=\frac{28}{3}\,\alpha Q^{-1}\left\{\frac{1}{2\pi\rho}(x^{11})^2-\frac{1}{2}\,x^{11}+\frac{\pi\rho}{12}\right\}\,, \tag{3.64}$$
$$\psi_2^{(X)}=-\frac{560}{3}\,\alpha Q^{-1}\left\{\frac{1}{2\pi\rho}(x^{11})^2-\frac{1}{2}\,x^{11}+\frac{\pi\rho}{12}\right\}\,, \tag{3.65}$$
$$\psi_{-2}^{(C)}=-\frac{3}{2}P_{-2}^{(f)}=\frac{182}{3}\alpha\,\frac{1}{\pi\rho}\,, \tag{3.66}$$
$$\psi_2^{(C)}=-168\,\alpha Q^{-1}\left\{\frac{1}{2\pi\rho}(x^{11})^2-\frac{1}{2}\,x^{11}+\frac{\pi\rho}{12}\right\}\,. \tag{3.67}$$

We note that, similarly to the $r^6\gg Q$ case, we need $\psi_2^{(C)}$ to obtain $\psi_2^{(X)}$.

Finally, we consider the x^{11}-independent part. From equations of motion, we find $A_1\sim B_1\sim X_1\sim C_1\sim r^{-2}$, and thus

$$A_1=\sum_{i=0}^{\infty}P_{-2+i}^{(A)}\,r^{-2+i}\,,\quad B_1=\sum_{i=0}^{\infty}P_{-2+i}^{(B)}\,r^{-2+i}\,,\quad X_1=\sum_{i=0}^{\infty}P_{-2+i}^{(X)}\,r^{-2+i}\,,$$
$$C_1=\sum_{i=0}^{\infty}P_{-2+i}^{(C)}\,r^{-2+i} \tag{3.68}$$

where $P_i^{(A)},P_i^{(B)},P_i^{(X)}$ and $P_i^{(C)}$ are constants. Expansions of equations are as follows.

The Maxwell equation:

$$\sum_i\,\Big[6i\Big\{-2P_i^{(A)}+6P_i^{(B)}-P_i^{(X)}\Big\}+i(i+6)P_i^{(C)}-6iP_i^{(f)}\Big]\,r^{i-2}=0\,,\qquad \underline{i\geq -2} \tag{3.69}$$

The Einstein equation

$$(\alpha,\beta):\quad \sum_i\,i(i+6)\Big\{-P_i^{(A)}+P_i^{(X)}\Big\}\,r^{i-2}=\frac{1}{2}\,\tilde{j}_1\,\alpha\,r^{-4}\,\frac{1}{\pi\rho}\,,\qquad \underline{i\geq -2} \tag{3.70}$$
$$(m,n):\quad \sum_i\,\Big[(36+6i)\Big\{6P_i^{(A)}+3P_i^{(X)}-3P_i^{(C)}\Big\}+i(i+6)\Big\{14P_i^{(A)}+42P_i^{(B)}+7P_i^{(X)}\Big\}\Big]\,r^{i-2}$$
$$=\tilde{j}_2\,\alpha\,r^{-4}\,\frac{1}{\pi\rho}\,,\qquad \underline{i\geq -2} \tag{3.71}$$
$$(11,11):\quad \sum_i\,(i+6)\Big\{2iP_i^{(A)}+(7i-18)P_i^{(B)}\Big\}\,r^{i-2}=0\,,\qquad \underline{i\geq -2} \tag{3.72}$$

From these equations, we obtain

$$P^{(A)}_{-2} = -8P^{(B)}_{-2} = -\frac{14}{15}\,\alpha\,\frac{1}{\pi\rho}\ , \quad P^{(X)}_{-2} = \frac{287}{30}\,\alpha\,\frac{1}{\pi\rho}\ , \quad P^{(C)}_{-2} = -\frac{301}{6}\,\alpha\,\frac{1}{\pi\rho}\ . \tag{3.73}$$

Consequently, from the above results (3.64)–(3.66) and (3.73), the solution is given by

$$\begin{aligned}
&A_1 = -8B_1 = -\frac{14}{15}\,\alpha\, r^{-2}\,\frac{1}{\pi\rho}\ , \qquad X_1 = \frac{287}{30}\,\alpha\, r^{-2}\,\frac{1}{\pi\rho}\ ,\\
&C_1 = -\frac{301}{6}\,\alpha\, r^{-2}\,\frac{1}{\pi\rho}\ , \qquad f = -\frac{364}{9}\,\alpha\, r^{-2}\,\frac{1}{\pi\rho}\ ,\\
&\phi_A = \phi_B = \frac{28}{3}\,\alpha Q^{-1}\, r^2 \left\{\frac{1}{2\pi\rho}(x^{11})^2 - \frac{1}{2}\,x^{11} + \frac{\pi\rho}{12}\right\}\ ,\\
&\phi_X = -\frac{560}{3}\,\alpha Q^{-1}\, r^2 \left\{\frac{1}{2\pi\rho}(x^{11})^2 - \frac{1}{2}\,x^{11} + \frac{\pi\rho}{12}\right\}\ ,\\
&\phi_C = \frac{182}{3}\,\alpha\, r^{-2}\,\frac{1}{\pi\rho}\ .
\end{aligned} \tag{3.74}$$

In two asymptotic regions $r^6 \gg Q$ and $r^6 \ll Q$, we have found solutions (3.58) and (3.74). In an intermediate region $r^6 \sim Q$, on the other hand, it is not clear how the solution behaves. However, we can solve equations of motion (3.5)–(3.9) numerically by using (3.58) and (3.74) as boundary conditions. In appendix, we plot $\Phi_{1A} = A_1 + \phi_A$ etc. as functions of x^{11} and r. The results become smooth surfaces, and thus it is certain that (3.58) and (3.74) describe a smooth solution of equations (3.5)–(3.9).

4 Interpretations of the solution

We have found the M2-brane solution of heterotic M-theory by solving equations of motion up to order $\kappa^{2/3}$ asymptotically in power series in r or r^{-1}. We discuss interpretations of this solution.

From the results of the previous section, we obtain the metric up to order $\kappa^{2/3}$. From (3.1), we recall that

$$g_{\alpha\beta} = e^{2A}\,\eta_{\alpha\beta} = \exp\Big[2\Big\{\,\frac{1}{3}\,Y_0 + \kappa^{2/3}\Phi_{1A} + \mathrm{O}(\kappa^{4/3})\Big\}\Big]\,\eta_{\alpha\beta} \tag{4.1}$$

where $\Phi_{1A} = \phi_A + A_1$ and so on. Then, from (3.58) and (3.74), we find that

<u>$r^6 \gg Q$</u>

$$\begin{aligned}
g_{\alpha\beta} &= \left[\,1 + 2\kappa^{2/3}\alpha\, Q^2 \left\{\frac{1568}{3}\, r^{-16}\, W(x^{11}) - \frac{14}{3}\, r^{-14}\,\frac{1}{\pi\rho}\right\} + \mathrm{O}(\kappa^{4/3})\right]\eta_{\alpha\beta}\ ,\\
g_{mn} &= \left[\,1 + 2\kappa^{2/3}\alpha\, Q^2 \left\{-\frac{448}{3}\, r^{-16}\, W(x^{11}) + \frac{4}{3}\, r^{-14}\,\frac{1}{\pi\rho}\right\} + \mathrm{O}(\kappa^{4/3})\right]\delta_{mn}\ ,\\
g_{11,11} &= \left[\,1 + 2\kappa^{2/3}\alpha\, Q^2 \left\{-\frac{448}{3}\, r^{-16}\, W(x^{11}) + \frac{4}{3}\, r^{-14}\,\frac{1}{\pi\rho}\right\} + \mathrm{O}(\kappa^{4/3})\right]\ ,
\end{aligned} \tag{4.2}$$

<u>$r^6 \ll Q$</u>

$$g_{\alpha\beta} = r^4 Q^{-2/3}\left[\,1 + 2\kappa^{2/3}\alpha \left\{\frac{28}{3}Q^{-1}\, r^2\, W(x^{11}) - \frac{14}{15}\, r^{-2}\,\frac{1}{\pi\rho}\right\} + \mathrm{O}(\kappa^{4/3})\right]\eta_{\alpha\beta}\ ,$$

$$g_{mn} = r^{-2}Q^{1/3}\left[\,1 + 2\kappa^{2/3}\alpha\left\{\frac{28}{3}Q^{-1}\,r^2\,W(x^{11}) + \frac{7}{60}\,r^{-2}\,\frac{1}{\pi\rho}\right\} + \mathrm{O}(\kappa^{4/3})\right]\delta_{mn}\;, \tag{4.3}$$
$$g_{11,11} = r^4Q^{-2/3}\left[\,1 + 2\kappa^{2/3}\alpha\left\{-\frac{560}{3}Q^{-1}\,r^2\,W(x^{11}) + \frac{287}{30}\,r^{-2}\,\frac{1}{\pi\rho}\right\} + \mathrm{O}(\kappa^{4/3})\right]$$

where

$$W(x^{11}) \equiv \frac{1}{2\pi\rho}(x^{11})^2 - \frac{1}{2}\,x^{11} + \frac{\pi\rho}{12}\;. \tag{4.4}$$

From the eleven-dimensional point of view, at order κ^0, this is the usual M2-brane solution of eleven-dimensional supergravity, but it receives the correction of order $\kappa^{2/3}$. In particular, it has a non-trivial x^{11}-dependence $W(x^{11})$ similar to the gauge 5-brane solution [11]. This correction has its origin in the source terms which consist of the Gauss-Bonnet terms in equations of motion (2.4), and their contributions, (3.13) and (3.14), are given by the non-trivial metric (3.1). The Gauss-Bonnet terms, on the other hand, are required by the cancellation of gravitational anomaly and supersymmetry [10, 4]. This anomaly is caused by the Z_2 singularities of the orbifold [17, 1, 2]. Therefore, the correction is regarded as a gravitational effect of the Z_2 singularities. If we are far away from the M2-brane ($r \to \infty$), A, B, X and C become zero. Thus we see that the metric becomes flat Minkovski and the correction vanishes.

Now, we discuss the solution from the viewpoint of ten-dimensional string theory. By the suggestion of Hořava and Witten [1], this solution is interpreted as the fundamental string solution of strongly coupled heterotic theory at low-energy. It is also suggested that the strong coupling limit corresponds to large radius of the orbifold. Then, the x^{11}-dependence which we have found explicitly must be the strong coupling correction. To check this phenomenon, we consider the string coupling constant. When we compactify M-theory on $S^1 \times M^{10}$, the string coupling constant g_s and the Regge slope α' are written by κ and R_{11}, the radius of the S^1 [22]. If we consider the S^1/Z_2 compactification, it is necessary to regard R_{11} as the volume of S^1/Z_2 given by $\int(g_{11,11})^{1/2}\,dx^{11}$ because of the x^{11}-dependence, and thus we have

$$g_s^2 = 2\pi^2\left(\frac{\kappa}{4\pi}\right)^{-2/3}\left(\int_0^{\pi\rho} e^X\,dx^{11}\right)^3 = \alpha'^{-1}\left(\int_0^{\pi\rho} e^X\,dx^{11}\right)^2\;. \tag{4.5}$$

If we consider κ as a unit, g_s^2 is given as, for $r^6 \gg Q$:

$$g_s^2 = 2\pi^2\left(\frac{\kappa}{4\pi}\right)^{-2/3}\left[(\pi\rho)^3 - 52\alpha\,\kappa^{2/3}Q^2\,r^{-14}(\pi\rho)^2\right] \tag{4.6}$$

and for $r^6 \ll Q$:

$$g_s^2 = 2\pi^2\left(\frac{\kappa}{4\pi}\right)^{-2/3}\left[Q^{-1}r^6(\pi\rho)^3 + \frac{287}{10}\,\alpha\,\kappa^{2/3}Q^{-1}r^4(\pi\rho)^2\right]\;. \tag{4.7}$$

As expected, we see that $\rho \ll 1$ leads to weak coupling limit, while $\rho \gg 1$ leads to strong coupling limit.

5 Conclusion

In this paper, we have considered the M2-brane solution of heterotic M-theory with the Gauss-Bonnet R^2 terms up to order $\kappa^{2/3}$. We derived equations of motion with source terms of order $\kappa^{2/3}$. By

the ansatz, only the Gauss-Bonnet terms contributed to source terms and the equations became non-linear which required κ expansion. We solved them asymptotically by expanding fields in r. The result is the usual M2-brane solution at order κ^0, but it receives a correction of order $\kappa^{2/3}$ which, in particular, has the x^{11}-dependence in the same form as the gauge 5-brane solution [11]. In appendix, we plot the solution as a function of x^{11} and r. It confirms that the asymptotic solutions connected smoothly.

From the eleven-dimensional point of view, this $\kappa^{2/3}$ correction can be regarded as the gravitational effect of an orbifold with Z_2 singularities, because the Gauss-Bonnet terms are required by the cancellation of the gravitational anomaly and supersymmetry, and only the metric contributes to the source terms through the Gauss-Bonnet terms in our case.

We discussed interpretations of this solution as the strongly coupled fundamental heterotic string at low-energy. Integrating this solution over x^{11}, we saw the expected behavior of the string coupling constant with a radius of the orbifold.

On the other hand, we have found the x^{11}-dependence explicitly which must have plenty of new information. Since it describes the strong coupling effect of string theory, it is very significant to analyze this solution with the x^{11}-dependence as a background solution in strongly coupled string theory or M-theory itself. We hope to find new applications of this solution. In addition, it is interesting to examine how this solution preserves supersymmetry with the correction of order $\kappa^{2/3}$ explicitly.

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Appendix

We solve equations of motion (3.5)–(3.9) numerically by using (3.58) and (3.74) as boundary conditions. For example, we choose $\rho = 1$, $Q = 100$ and plot $\Phi_{1A} = A_1 + \phi_A$ etc. in (3.2) and (3.4) as functions of x^{11} and $r = (x^m x^n \delta_{mn})^{1/2}$. The range of the x^{11}- and r-axes are $[0, \pi]$ and $[R_0, R]$ where $R_0 = (10^{-1}Q)^{1/6} \sim 1.47$ and $R = (10\,Q)^{1/6} \sim 3.16$. These figures indicate that the asymptotic solutions, (3.58) and (3.74), are connected smoothly.

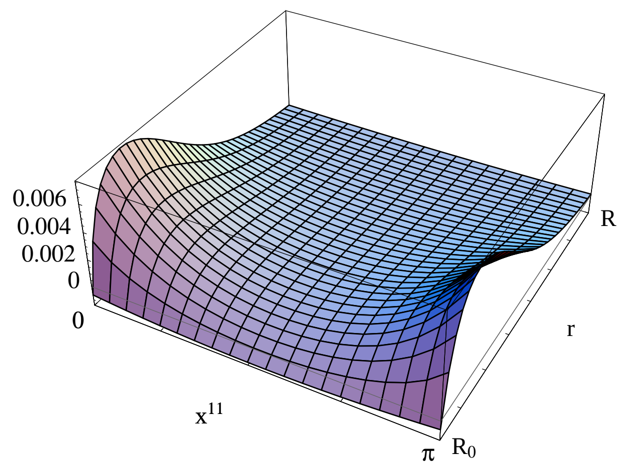

Figure 2: Φ_{1A}

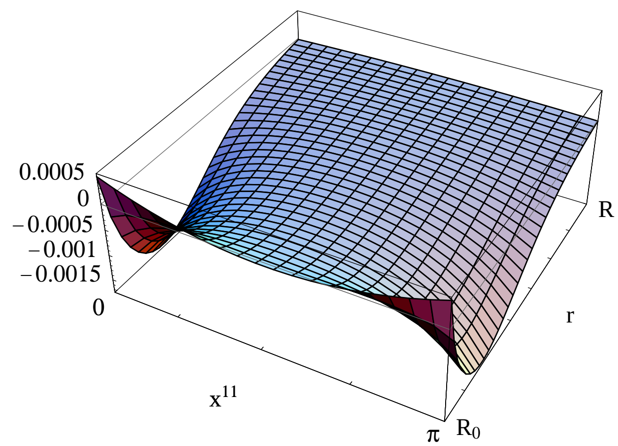

Figure 3: Φ_{1B}

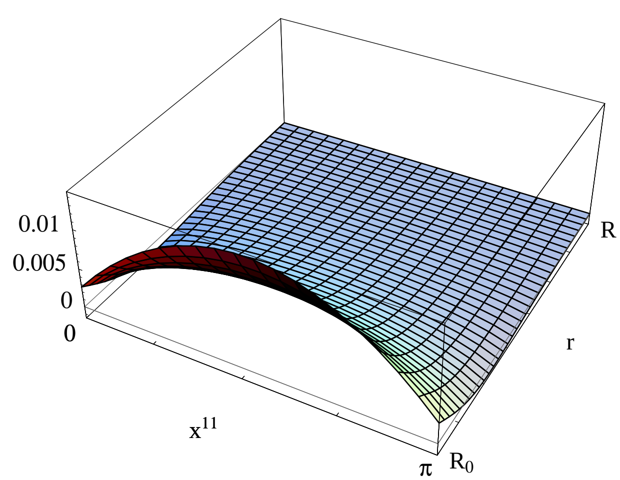

Figure 4: Φ_{1X}

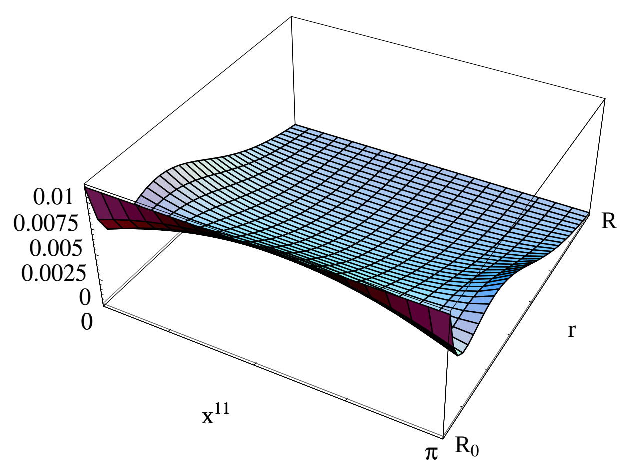

Figure 5: Φ_{1C}

The $\kappa^{2/3}$ correction from the Gauss-Bonnet terms

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